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On generalized Nambu mechanics

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Abstract. A geometric formulation of a generalization of Nambu mechanics is proposed. This formulation is carried out, wherever possible, in analogy with that of Hamiltonian systems. In this formulation, a strictly non-degenerate constant 3-form is attached to a $3n$-dimensional phase space. Time evolution is governed by two Nambu functions. A Poisson bracket of 2-forms is introduced, which provides a Lie-algebra structure on the space of 2-forms. This formalism is shown to provide a suitable framework for the description of non-integrable fluid flow such as Arter flow, Chandrashekhar flow and of coupled rigid bodies.

1. Introduction

In 1973, Nambu proposed a generalization of Hamiltonian mechanics by considering systems which obey the Liouville theorem in three-dimensional phase space [1]. In this formalism, the points of the phase space were labelled by a canonical triplet $\tilde{r} = (x, y, z)$. A pair of Hamiltonian-like functions H_1, H_2, (which we call Nambu functions hereafter), were introduced on this phase space. In terms of these functions the equations of motion were written as

$$\frac{d\tilde{r}}{dt} = \tilde{\nabla} H_1 \times \tilde{\nabla} H_2. \tag{1}$$

Nambu also defined a generalization of Poisson bracket on this new phase space by

$$\{F, H_1, H_2\} = \tilde{\nabla} F \cdot (\tilde{\nabla} H_1 \times \tilde{\nabla} H_2). \tag{2}$$

An attempt was made by him to find a quantized version of the formalism, but he succeeded only partially, since the correspondence between the classical and quantum versions is largely lost [1].

The possibility of embedding the dynamics of a Nambu triplet in a four-dimensional canonical phase-space formalism was proved in [2, 3], but such an embedding is local and non-unique.

An algebraic approach, which was suitable for quantization, was developed in [4, 5] where a generalization of the Nambu bracket was postulated. In this approach a rather rigid consistency condition, called the fundamental identity, which is a generalization of the Jacobi identity for Poisson brackets, was introduced. The algebraic approach, no doubt, is quite elegant but is too restrictive, in the sense that the dynamics on a n-dimensional manifold is determined by $(n-1)$ functions $H_1, \ldots, H_{n-1}$ which are integrals of motion.

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Due to this large number of integrals of motion, this formalism is not suitable for the formulation of non-integrable or chaotic systems.

The geometric formulation of Hamiltonian mechanics has revealed several deep insights. One would expect that similar insight would emerge from geometric formulation of Nambu systems. Such possibilities were first examined by Estabrook [6] and more recently by Fecko [7].

Recall that in the formulation of Hamiltonian systems, there exist several equivalent ways [8] of endowing an even-dimensional manifold M^{2n} with a symplectic structure. Two of the main ways are to

(i) attach a closed non-degenerate 2-form $\omega^{(2)}$;
(ii) attach a bracket on the class of C^{∞} functions on M^{2n} with properties of bilinearity, skew-symmetry, Leibnitz rule, Jacobi identity and non-degeneracy (i.e. a Poisson structure together with non-degeneracy).

Both types of approaches have been tried with Nambu systems [1–7]:

- As mentioned above, it was found that the algebraic approach, starting with bracket of functions, required the introduction of a rather rigid condition in the form of the fundamental identity [5, 9] for consistency.
- On the other hand, the geometric analysis [6, 7] led to the conclusion that it was impossible to obtain a volume form from a 3-form, except in the most trivial cases.

The complications mentioned above arise in both cases, because volume preservation is a very stringent requirement. In the light of these findings we propose in this paper that one need not impose volume preservation to construct a geometric formalism of Nambu systems.

In this paper, we restrict ourselves to what we call a Nambu system of order 3. Such a system has a $3n$-dimensional phase space and two Nambu functions. We introduce a Nambu manifold as a $3n$-dimensional manifold M^{3n}, together with a constant 3-form $\omega^{(3)}$ which is strictly non-degenerate. (The notion of non-degeneracy of 2-forms requires modification. This modified notion is called strict non-degeneracy.) There is a natural generalization of the Darboux basis corresponding to Hamiltonian systems. Equations of motion (Nambu equations) are introduced in terms of two Nambu functions which are analogous to the Hamiltonian function in Hamiltonian dynamics. A novel feature of the present paper is that there is a natural way of introducing the Nambu–Poisson bracket of 2-forms. The form $\omega^{(3)}$ is preserved in the present approach which may be compared with the preservation of $\omega^{(2)}$ for a Hamiltonian system. In Hamiltonian dynamics all powers of $\omega^{(2)}$ are canonical invariants. However, in the present formalism, since $\omega^{(3)}$ is an odd-order form, no conclusions about canonical invariants can be obtained from the preservation of $\omega^{(3)}$ itself.

The real justification for such a generalization can emerge from application to realistic physical systems and from better algorithmic strategies. We demonstrate that the non-integrable Arter flow and Chandrashekhar flow describing Rayleigh–Bénard convective motion with rotation can suitably be described in our framework. We note further that the algorithmic strategy developed in [11] can now be identified as a generalization (in our framework) of symplectic integration corresponding to the Hamiltonian framework.

In section 2 we develop the geometric formulation. In section 2.1, The notion of strict non-degeneracy of 3-forms is introduced. The notion of Nambu vector space is defined using strict non-degeneracy. The existence of a Darboux-like basis is proved. This is followed by the notion of a Nambu map. In section 2.2 a Nambu manifold of order 3 and canonical transformations on this manifold are defined. Furthermore, a correspondence between 2-forms and vectors is established. This is followed by a discussion of conditions under which the 3-form $\omega^{(3)}$ is preserved. In section 2.3 the Nambu systems are defined. A Nambu vector field corresponding to two Nambu functions is introduced. It is proved that the phase flow preserves the Nambu structure. In section 2.4 a bracket of 2-forms is introduced. This bracket provides a Lie algebra structure on the space of 2-forms.

In section 3 concrete applications of this framework to the examples of coupled rigid bodies and to the fluid flows are described.

2. Geometric formulation of Nambu systems

We begin by recalling the essential features of the Hamiltonian formalism. The phase space has the structure of a smooth manifold M. A closed, non-degenerate 2-form, namely the symplectic 2-form $\omega^{(2)}$, is attached to this manifold. The non-degeneracy of $\omega^{(2)}$ imposes the condition that M be even dimensional. Canonical transformations are those transformations under which the 2-form $\omega^{(2)}$ remains invariant. Use of $\omega^{(2)}$ allows us to establish an isomorphism between 1-forms and vector fields. The time evolution is governed by the Hamiltonian vector field X_H which is simply the vector field associated with the 1-form dH, where H is a smooth function on M.

Alternatively one can introduce the Poisson brackets on the space of C^∞ functions on M. The Poisson bracket, together with the non-degeneracy condition, induces a symplectic structure.

In the present paper, wherever possible, we develop the framework of Nambu systems in analogy with that of Hamiltonian systems.

2.1. Nambu vector space

In this section we define the Nambu vector space which is analogous to the symplectic vector space in Hamiltonian mechanics. The Nambu vector space is a vector space with a strictly non-degenerate 3-form. We prove that in such a space there exists a preferred choice of basis which we call the Nambu–Darboux basis.

Definition 2.1 (non-degenerate form). Let E be a finite-dimensional vector space and let $\omega^{(3)}$ be a 3-form on E, i.e.

$$\omega^{(3)} : E \times E \times E \to \mathbb{R}.$$

Then the form $\omega^{(3)}$ is called a *non-degenerate form* if

$$\forall \text{ non-zero } e_1 \in E \;\; \exists \;\; e_2, e_3 \in E \qquad \text{such that } \; \omega^{(3)}(e_1, e_2, e_3) \neq 0.$$

Remark 2.1. In three dimensions every non-zero 3-form is non-degenerate.

Remark 2.2. If the dimension of the vector space is less than three, then one cannot have an anti-symmetric, non-degenerate 3-form.

Remark 2.3. A non-degenerate 3-form allows us to define an analogue of the orthogonal complement as follows.

Definition 2.2 (Nambu complement). Let E be an m-dimensional vector space with $m \geqslant 3$. Let $\omega^{(3)}$ be an anti-symmetric and non-degenerate 3-form on E. Let us choose $e_1, e_2, e_3 \in E$ such that $\omega^{(3)}(e_1, e_2, e_3) \neq 0$. Let $P_1 = \mathrm{Span}(e_1, e_2, e_3)$. Then the *Nambu complement* of P_1 is defined as

$$P_1^{\perp_E} = \{z \in E \;\mid\; \omega^{(3)}(z, z_1, z_2) = 0 \;\; \forall \;\; z_1, z_2 \in P_1\}.$$

Proposition 2.1. Let E be an m-dimensional vector space with $m \geqslant 3$. Let $\omega^{(3)}$ be an anti-symmetric and non-degenerate 3-form on E. Let us choose $e_1, e_2, e_3 \in E$ such that $\omega^{(3)}(e_1, e_2, e_3) \neq 0$. We further choose e_2, e_3 such that $\omega^{(3)}(e_1, e_2, e_3) = 1$. Let $P_1 = \mathrm{Span}(e_1, e_2, e_3)$. Then $E = P_1 \oplus P_1^{\perp_E}$.

Proof. We write $\forall\, x \in E$

$$x' = x - \omega^{(3)}(x, e_2, e_3)e_1 - \omega^{(3)}(x, e_3, e_1)e_2 - \omega^{(3)}(x, e_1, e_2)e_3.$$

It is easy to see that $x' \in P_1^{\perp_E}$. From the definition of $P_1^{\perp_E}$ it follows that $P_1 \cap P_1^{\perp_E} = \{0\}$. Hence $E = P_1 \oplus P_1^{\perp_E}$. □

Definition 2.3 (strictly non-degenerate form). Let E be an m-dimensional vector space and $\omega^{(3)}$ be an anti-symmetric and non-degenerate 3-form on E. Then $\omega^{(3)}$ is called *strictly non-degenerate* if for each non-zero $e_1 \in E$ $\exists$ a two-dimensional subspace $E_1 \subset E$ such that

(i) $\omega^{(3)}(e_1, x_1, x_2) \neq 0$ $\forall$ linearly independent $\{e_1, x_1, x_2\}$ where $x_1, x_2 \in F_1$ and $F_1 = \mathrm{Span}(e_1 + E_1)$.
(ii) $\omega^{(3)}(e_1, z_1, z_2) = 0$ $\forall$ $z_1, z_2 \in F_1^{\perp_E}$.

Remark 2.4. In three-dimensional space every non-degenerate form is strictly non-degenerate.

Definition 2.4 (Nambu vector space). Let E be a finite-dimensional vector space and $\omega^{(3)}$ be a completely anti-symmetric and strictly non-degenerate 3-form on E. Then the pair $(E, \omega^{(3)})$ is called a *Nambu vector space*.

Recall that the rank of a 2-form is defined as the rank of its matrix representation. We now introduce the notion of rank for the anti-symmetric 3-forms.

Definition 2.5 (rank of $\omega^{(3)}$). Let E be a finite-dimensional vector space and $\omega^{(3)}$ be a completely anti-symmetric 3-form. Then the *rank of* $\omega^{(3)}$ is defined as

$$\sup_{P \subset E}\{d \mid d = \dim P,\ (P, \omega^{(3)}|_P) \text{ is a Nambu vector space}\}.$$

Remark 2.5. The following proposition gives the prescription for constructing the Nambu–Darboux basis.

Proposition 2.2. Let E be an m-dimensional vector space. Let $\omega^{(3)}$ be a 3-form of rank m on E. Then $m = 3n$ for a unique integer n. Furthermore, there is an ordered basis $\{e_i\}, i = 1, \ldots, m$ with the corresponding dual basis $\{\alpha^i\}, i = 1, \ldots, m$, such that

$$\omega^{(3)} = \begin{cases} \displaystyle\sum_{i=0}^{n-1} \alpha^{3i+1} \wedge \alpha^{3i+2} \wedge \alpha^{3i+3} & \text{if } n > 0 \\ \omega^{(3)} = 0 & \text{otherwise.} \end{cases}$$

Proof. The rank of $\omega^{(3)}$ is m, implies that $(E, \omega^{(3)})$ is a Nambu vector space. One can assume that $m \geqslant 3$, for otherwise the proposition is trivial with $n = 0$. Let $e_1 \in E$ and let $E_1 \subset E$ be a 2-dimensional subspace such that $\omega^{(3)}(e_1, e_2, e_3) \neq 0$ $\forall$ linearly independent $\{e_1, e_2, e_3\}$ where $e_2, e_3 \in P_1$ and $P_1 = \mathrm{Span}(e_1 + E_1)$. Let e_2, e_3 be a basis of E_1, then e_1, e_2, e_3 is a basis of P_1. Thus $(P_1, \omega^{(3)}|_{P_1})$ is three-dimensional Nambu vector space. It follows that one can write $\omega^{(3)}|_{P_1}$ in basis $\alpha^1, \alpha^2, \alpha^3$ dual to e_1, e_2, e_3 as

$$\omega^{(3)}|_{P_1} = \alpha^1 \wedge \alpha^2 \wedge \alpha^3$$

If $m = 3$ then $P_1 = E$ and the proposition is proved with $n = 1$. Hence we assume $m > 3$. We denote $Q_1 = P_1^{\perp_E}$. The dimension of Q_1 is $m - 3$. Since the vector space E is a Nambu vector space it follows $\dim(Q_1) \geqslant 3$. Let $e_4 \in Q_1 \subset E \Rightarrow \exists\, e_5, e_6 \in E$ such that $\omega^{(3)}(e_4, e_5, e_6) \neq 0\ \forall$ linearly independent $\{e_4, e_5, e_6\}$ where $e_5, e_6 \in P_2$ and $P_2 = \mathrm{Span}(e_4, e_5, e_6)$. By definition of Q_1 it follows that $e_5, e_6 \notin P_1$. We choose $e_5, e_6 \in P_1^{\perp_E} = Q_1$. It follows from the strict non-degeneracy of $\omega^{(3)}$ in E and the facts $P_2^{\perp_{Q_1}} \subset P_2^{\perp_E}$ and $P_2 \subset Q_1$ that $(Q_1, \omega^{(3)}|_{Q_1})$ is a Nambu vector space of dimension $m - 3$.

Repeated application of the above argument on Q_1 in place of E and so on yields

$$E = P_1 \oplus \cdots \oplus P_n.$$

We stop the recursion when $\dim(Q_n) = 0$. Using the strict non-degeneracy of $\omega^{(3)}$ we get

$$\begin{aligned}\omega^{(3)} &= \omega^{(3)}|_{P_1} + \cdots + \omega^{(3)}|_{P_n} \\ &= \sum_{i=0}^{n-1} \alpha^{3i+1} \wedge \alpha^{3i+2} \wedge \alpha^{3i+3}.\end{aligned}$$ □

Definition 2.6 (Nambu mappings). If (E, ω) and (F, ρ) are two Nambu vector spaces and $f : E \to F$ is a linear map such that the pullback $f^*\rho = \omega$, then f is called a *Nambu mapping*.

Proposition 2.3. Let (E, ω) and (F, ρ) be two Nambu vector spaces of dimension $3n$ and let a linear map $f : E \to F$ be Nambu mapping. Then f is an isomorphism on the vector spaces.

Proof. Let if possible, f be singular. Then there exists $x \in E$ and $x \neq 0$ such that $fx = 0$. But since f is a Nambu mapping one can write

$$\rho(f(x), f(y), f(z)) = \omega(x, y, z)$$

where $y, z \in E$ are so chosen that $\omega(x, y, z) \neq 0$. This leads to a contradiction. Hence f is an isomorphism. □

Proposition 2.4. Let $(E, \omega^{(3)})$ be a Nambu vector space of dimension $3n$. Then the set of Nambu mappings from E to E forms a group under composition.

Proof. Now let f and g be Nambu mappings. Then

$$(f \circ g)^* \omega^{(3)} = g^* \circ f^* \omega^{(3)} = g^* \omega^{(3)} = \omega^{(3)}$$

and

$$(f^{-1})^* \omega^{(3)} = (f^*)^{-1} \omega^{(3)} = \omega^{(3)}.$$ □

2.2. *Nambu manifold*

In analogy with the notion of symplectic manifold we now introduce Nambu manifold.

Definition 2.7 (Nambu manifold). Let M^{3n} be a $3n$-dimensional C^∞ manifold and let $\omega^{(3)}$ be a 3-form field on M^{3n} such that $\omega^{(3)}$ is completely anti-symmetric, constant (i.e. a constant section on the bundle of 3-forms) and strictly non-degenerate at every point of M^{3n}. Then the pair $(M^{3n}, \omega^{(3)})$ is called a *Nambu manifold*.

Remark 2.6. In the case of Hamiltonian systems the form $\omega^{(2)}$ is assumed to be closed, here $\omega^{(3)}$ is assumed to be a constant form. (Whether or not a constant $\omega^{(3)}$ is admissible on a manifold is a topological issue. This restriction would not allow us to consider some manifolds as Nambu manifolds.) The condition of closedness allows many 3-forms which in general may not be consistent with the non-degeneracy condition.

Remark 2.7. In fact it is possible to relax the condition of constancy of $\omega^{(3)}$ to the condition of $\omega^{(3)}$ being locally constant.

Theorem 2.1 (Nambu–Darboux theorem). Let $(M^{3n}, \omega^{(3)})$ be a Nambu manifold. Then at every point $p \in M^{3n}$ there is a chart (U, ϕ) in which $\omega^{(3)}$ is written as

$$\omega^{(3)}|_U = \sum_{i=0}^{n-1} dx_{3i+1} \wedge dx_{3i+2} \wedge dx_{3i+3}$$

where $(x_1, x_2, x_3, \ldots, x_{3(n-1)+1}, x_{3(n-1)+2}, x_{3(n-1)+3})$ are local coordinates on U described by ϕ.

Proof. Since $\omega^{(3)}$ is a constant 3-form, on every chart we can use the proposition 2.2 and get the required form for $\omega^{(3)}$. □

The coordinates described in theorem 2.1 will be called Nambu–Darboux coordinates hereafter. We use these coordinates in the remaining parts of the paper.

Definition 2.8 (canonical transformation). Let $(M^{3n}, \omega^{(3)})$ and $(N^{3n}, \rho^{(3)})$ be Nambu manifolds. A C^∞ mapping $F : M^{3n} \to N^{3n}$ is called a *canonical transformation* if $F^*\rho^{(3)} = \omega^{(3)}$.

Let $\mathcal{T}_k^0(M^{3n})$ denote a bundle of k-forms on M^{3n}, $\Omega_k^0(M^{3n})$ the space of k-form fields on M^{3n} and $\mathcal{X}(M^{3n})$ the space of vector fields on M^{3n}. Now for a given vector field X on M^{3n} we denote

$$i_X : \Omega_k^0(M^{3n}) \to \Omega_{k-1}^0(M^{3n})$$

as the inner product of X, with the k-form or contraction of a k-form by X being given by

$$(i_X \eta^{(k)})(a_1, \ldots, a_{k-1}) = \eta^{(k)}(X, a_1, \ldots, a_{k-1})$$

where $\eta^{(k)} \in \Omega_k^0(M^{3n})$ and $a_1, \ldots, a_{k-1} \in \mathcal{X}(M^{3n})$.

We now define the analogues of the raising and lowering operations. The map $\flat : \mathcal{X}(M^{3n}) \to \Omega_2^0(M^{3n})$ is defined by $X \mapsto X^\flat = i_X\omega^{(3)}$. In contrast, the map $\sharp : \Omega_2^0(M^{3n}) \to \mathcal{X}(M^{3n})$, is defined by the following prescription. Let α be a 2-form and α_{ij} be its components in Nambu–Darboux coordinates. Then the components of $\alpha^\sharp$ are given by

$$\alpha^{\sharp\, 3i+p} = \frac{1}{2} \sum_{l,m=1}^{3} \varepsilon_{plm} \alpha_{3i+l\ \ 3i+m} \tag{3}$$

where $0 \leqslant i \leqslant n-1$, $p = 1, 2, 3$ and ε_{plm} is the Levi–Cività symbol.

Remark 2.8. It may appear that components of $\alpha^\sharp$ have been given a definition using a particular choice of coordinate system. The definition itself is actually coordinate free, as shown in the appendix.

Remark 2.9. In contrast to the customary meaning of $\flat$ and $\sharp$ used in ordinary tensor analysis, we note that in this paper $\flat$ maps a vector to a 2-form and not to a 1-form; also $\sharp$ maps a 2-form to a vector. From the above definition it is clear that $(X^\flat)^\sharp = X$ but $(\alpha^\sharp)^\flat$ may not always yield the same α.

Remark 2.10. In fact consider $\mathcal{T}^0_{2_x}(M^{3n})$, the space of 2-forms at $x \in M^{3n}$. The $\sharp$ defines an equivalence relation on $\mathcal{T}^0_{2_x}(M^{3n})$ as follows. Let $\omega_1^{(2)}(x), \omega_2^{(2)}(x) \in \mathcal{T}^0_{2_x}(M^{3n})$. We say that $\omega_1^{(2)}(x) \sim \omega_2^{(2)}(x)$ if $(\omega_1^{(2)})^\sharp(x) = (\omega_2^{(2)})^\sharp(x)$. It is easy to see that $\sim$ is an equivalence relation. We define the equivalence classes $\mathcal{S}^0_{2_x}(M^{3n}) = \mathcal{T}^0_{2_x}(M^{3n})/\sim$.

Let $\omega_1^{(2)}(x), \omega_2^{(2)}(x), \omega_3^{(2)}(x) \in \mathcal{T}^0_{2_x}(M^{3n})$ and the equivalence class be denoted by [ ], i.e. $\alpha_1(x) = [\omega_1^{(2)}(x)], \alpha_2(x) = [\omega_2^{(2)}(x)], \alpha_3(x) = [\omega_3^{(2)}(x)]$ where $\alpha_1(x), \alpha_2(x), \alpha_3(x) \in \mathcal{S}^0_{2_x}(M^{3n})$. The addition and scalar multiplication on $\mathcal{S}^0_{2_x}(M^{3n})$ are defined as follows:

$$\alpha_1(x) + \alpha_2(x) = [\omega_1^{(2)}(x) + \omega_2^{(2)}(x)]$$

$$\mu \cdot \alpha_1(x) = [\mu \cdot \omega_1^{(2)}(x)]$$

where $\mu \in \mathbb{R}$. It is easy to see that $(\mathcal{S}^0_{2_x}(M^{3n}), +, \mathbb{R}, \cdot)$ forms a vector space and that the dimension of this vector space is $3n$.

Now we investigate conditions under which the given Nambu form $\omega^{(3)}$ is invariant under the action of the vector field $\beta^\sharp$ associated with any 2-form β.

Proposition 2.5. Let $\beta^{(2)} \in \Omega^0_2(M^{3n})$. Then $(\beta^{(2)^\sharp})^\flat \sim \beta^{(2)}$.

Proof. The proof follows from the fact that $(X^\flat)^\sharp = X \ \ \forall X \in \mathcal{X}(M^{3n})$. □

Theorem 2.2. Let $\beta^{(2)} \in \Omega^0_2(M^{3n})$, and f^t be a flow corresponding to $\beta^{(2)^\sharp}$, i.e. $f^t : M^{3n} \to M^{3n}$ such that

$$\frac{d}{dt}\bigg|_{t=0}(f^t x) = (\beta^{(2)^\sharp})x \qquad \forall x \in M^{3n}.$$

Then the form $\omega^{(3)}$ is preserved under the action of $\beta^{(2)^\sharp}$ iff $d(\beta^{(2)^\sharp})^\flat = 0$, i.e. $f^{t*}\omega^{(3)} = \omega^{(3)}$ iff $d(\beta^{(2)^\sharp})^\flat = 0$

Proof. We have

$$\begin{aligned}\frac{d}{dt}(f^{t*}\omega^{(3)}) &= f^{t*}(L_{\beta^{(2)\sharp}}\omega^{(3)}) \\ &= f^{t*}(i_{\beta^{(2)\sharp}}d\omega^{(3)} + d(i_{\beta^{(2)\sharp}}\omega^{(3)})) \\ &= f^{t*}d(\beta^{(2)^\sharp})^\flat.\end{aligned}$$

If $d(\beta^{(2)^\sharp})^\flat = 0 \Rightarrow \frac{d}{dt}(f^{t*}\omega^{(3)}) = 0$ and also if $\frac{d}{dt}(f^{t*}\omega^{(3)}) = 0 \Rightarrow d(\beta^{(2)^\sharp})^\flat = 0$. □

From the above theorem it follows that the vector field corresponding to a 2-form preserves the Nambu structure if that 2-form is equivalent to a closed 2-form. By the Poincaré lemma one can locally write the closed 2-form as $d\xi$ where ξ is a 1-form. Without loss of generality we choose $\xi = f_1\, df_2$ where f_1 and f_2 are C^∞ functions on M^{3n}. So we can choose $(\beta^{(2)^\sharp})^\flat$ as $df_1 \wedge df_2$ and by proposition 2.5 $(df_1 \wedge df_2) \sim \beta^{(2)}$.

2.3. *The Nambu system*

Having introduced the relevant structure, namely the Nambu manifold, we now proceed with the discussion of time-evolution. The time-evolution is governed by a vector field obtained from two Nambu functions.

Definition 2.9 (Nambu vector field). Let $\mathcal{H}_1$, $\mathcal{H}_2$ be real-valued C^∞ functions (Nambu functions) on $(M^{3n}, \omega^{(3)})$. Then N is called the *Nambu vector field* corresponding to $\mathcal{H}_1, \mathcal{H}_2$ if

$$N = (d\mathcal{H}_1 \wedge d\mathcal{H}_2)^\sharp.$$

Definition 2.10 (Nambu system). A quadruple $(M^{3n}, \omega^{(3)}, \mathcal{H}_1, \mathcal{H}_2)$ is called a *Nambu system*.

Henceforth we choose $d\mathcal{H}_1 \wedge d\mathcal{H}_2$ as the representative 2-form of the class $[d\mathcal{H}_1 \wedge d\mathcal{H}_2]$. We note that in the equivalence class of $d\mathcal{H}_1 \wedge d\mathcal{H}_2$ there are some 2-forms which are not closed and cannot be expressed as $d\mathcal{H}_1 \wedge d\mathcal{H}_2$.

Now for a given representative 2-form we have freedom in the choice of $\mathcal{H}_1$ and $\mathcal{H}_2$. This freedom is discussed in [1] as gauge freedom in the choice of $\mathcal{H}_1$ and $\mathcal{H}_2$. The gauge freedom is quite different from the freedom in the definition of $\mathcal{S}_2^0$ and is a generalization of the freedom of the additive constant in Hamiltonian dynamics.

Definition 2.11 (Nambu phase flow). Let $(M^{3n}, \omega^{(3)}, \mathcal{H}_1, \mathcal{H}_2)$ be a Nambu system. Then the set of diffeomorphisms $g^t : M^{3n} \to M^{3n}$ satisfying

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} (g^t \boldsymbol{x}) &= (d\mathcal{H}_1 \wedge d\mathcal{H}_2)^\sharp \boldsymbol{x} \qquad \forall \boldsymbol{x} \in M^{3n} \\ &= N\boldsymbol{x} \end{aligned}$$

is called the *Nambu phase flow*.

From the flow properties of a differentiable vector field it follows that g^t is a one-parameter group of diffeomorphisms.

Theorem 2.3. Nambu phase flow preserves the Nambu structure, i.e.

$$g^{t*}\omega^{(3)} = \omega^{(3)}.$$

Proof. The proof follows from theorem 2.2. □

Remark 2.11. Since the proof of theorem 2.3 is valid for any time t provided the flow g^t exists, it is automatically valid for any time interval, say from t_1 to t_2. Furthermore, the flow preserves the $\omega^{(3)}$, which implies that the map $g^t : M^{3n} \to M^{3n}$ is a canonical transformation. This leads to the interpretation, as in the case of Hamiltonian systems, that *'The history of a Nambu system is a gradual unfolding of successive canonical transformation.'* Such an observation is one of the crucial ingredients required for the development of symplectic integrators for Hamiltonian systems. The present observation can therefore be used for a similar algorithm for Nambu systems.

Proposition 2.6. Let $(M^{3n}, \omega^{(3)}, \mathcal{H}_1, \mathcal{H}_2)$ be a Nambu system. Then $\mathcal{H}_1$ and $\mathcal{H}_2$ are constants of motion.

Proof. We prove the result for $\mathcal{H}_1$ (the proof for $\mathcal{H}_2$ is similar):

$$\begin{aligned} \frac{d}{dt}\mathcal{H}_1 &= L_N \mathcal{H}_1 \\ &= i_N d\mathcal{H}_1 \\ &= d\mathcal{H}_1(N) \\ &= d\mathcal{H}_1((d\mathcal{H}_1 \wedge d\mathcal{H}_2)^\sharp). \end{aligned}$$

If we write the right-hand side in Nambu–Darboux coordinates using equations (3) we get RHS = 0. □

2.4. The Nambu bracket

We now define the analogue of the Poisson bracket for 2-forms. This leads to the algebra of 2-forms. Furthermore, we also define the brackets for three functions in the conventional fashion [1].

Definition 2.12 (Nambu Poisson bracket). Let $\omega_a^{(2)}$ and $\omega_b^{(2)}$ be 2-forms. Then the *Nambu Poisson bracket* is a map $\{,\} : \Omega_2^0(M^{3n}) \times \Omega_2^0(M^{3n}) \to \Omega_2^0(M^{3n})$ given by

$$\{\omega_a^{(2)}, \omega_b^{(2)}\} = [\omega_a^{(2)\sharp}, \omega_b^{(2)\sharp}]^\flat$$

where [,] is the Lie bracket of vector fields.

Thus one writes

$$\{\omega_a^{(2)}, \omega_b^{(2)}\}(\xi, \eta) = (i_{[\omega_a^{(2)\sharp}, \omega_b^{(2)\sharp}]}\omega^{(3)})(\xi, \eta) \qquad \forall\, \xi, \eta \in \mathcal{X}(M^{3n}).$$

From the definition it clear that if $\omega_a^{(2)} \sim \omega_b^{(2)}$ then the $\{\omega_a^{(2)}, \omega_b^{(2)}\} = 0$ and if $\omega_a^{(2)} \sim \omega_{a'}^{(2)}$ and $\omega_b^{(2)} \sim \omega_{b'}^{(2)}$ then $\{\omega_a^{(2)}, \omega_b^{(2)}\} = \{\omega_{a'}^{(2)}, \omega_{b'}^{(2)}\}$.

Proposition 2.7. Let $(M^{3n}, \omega^{(3)})$ be a Nambu manifold and $\alpha, \beta \in \Omega_2^0(M^{3n})$ then

$$\{\alpha, \beta\} = L_{\alpha^\sharp}(\beta^\sharp)^\flat - L_{\beta^\sharp}(\alpha^\sharp)^\flat - d(i_{\alpha^\sharp} i_{\beta^\sharp}\omega^{(3)}).$$

Proof. We have

$$\begin{aligned}\{\alpha, \beta\} &= i_{[\alpha^\sharp, \beta^\sharp]}\omega^{(3)}\\ &= L_{\alpha^\sharp}(i_{\beta^\sharp}\omega^{(3)}) - i_{\beta^\sharp}(L_{\alpha^\sharp}\omega^{(3)})\\ &= L_{\alpha^\sharp}(\beta^\sharp)^\flat - i_{\beta^\sharp} d(i_{\alpha^\sharp}\omega^{(3)})\\ &= L_{\alpha^\sharp}(\beta^\sharp)^\flat - L_{\beta^\sharp}(\alpha^\sharp)^\flat - d(i_{\alpha^\sharp} i_{\beta^\sharp}\omega^{(3)}).\end{aligned}$$ □

Proposition 2.8. Let α and β be 2-forms. Further let $\alpha^\sharp$ be a Nambu vector field. Let $\alpha' = (\alpha^\sharp)^\flat$ and $\beta' = (\beta^\sharp)^\flat$ then

$$\{\alpha, \beta\} = L_{\alpha'^\sharp}\beta'.$$

Proof. The proof follows from proposition 2.7. □

Proposition 2.9. The space $\Omega_2^0(M^{3n})$ forms a Lie algebra with multiplication defined by the bracket, i.e. if $\alpha, \beta, \gamma \in \Omega_2^0(M^{3n})$

(i) $\{\alpha + \gamma, \beta\} = \{\alpha, \beta\} + \{\gamma, \beta\}$ and $\{\alpha, \beta + \gamma\} = \{\alpha, \beta\} + \{\alpha, \gamma\}$;
(ii) $\{\alpha, \alpha\} = 0$;
(iii) $\{\alpha, \{\beta, \gamma\}\} + \{\beta, \{\gamma, \alpha\}\} + \{\gamma, \{\alpha, \beta\}\} = 0$.

Proof. This follows from the definition of the bracket. □

Remark 2.12. In the Hamiltonian systems smooth functions on the phase space are considered as observables. To each such function f a natural vector field (namely X_f which is in correspondence with df) is attached. We note that it is really to the 1-form df that a vector field is attached (all functions differing by constants form an equivalence class producing identical df.)

In view of the above discussion it is clear that in the Nambu framework 2-forms play a basic role. In this connection we point out the following facts:

(i) Vector fields are naturally associated with 2-forms.

(ii) The bracket of 2-forms provides the Lie algebra structure. On the other hand bracket of functions introduces a non-associative structure as noted by [1, 9].

Definition 2.13 (Nambu bracket for functions). Consider a Nambu manifold $(M^{3n}, \omega^{(3)})$ and let f, g, h be C^∞ functions on M^{3n}. Then the *Nambu bracket for functions* is given by

$$\{f, g, h\} = L_{(dg\wedge dh)^\sharp} f = i_{(dg\wedge dh)^\sharp} df.$$

By using equations (3) in Nambu–Darboux coordinates

$$\{f, g, h\} = \sum_{i=0}^{n-1} \sum_{k,l,m=1}^{3} \varepsilon_{klm} \frac{\partial f}{\partial x_{3i+k}} \frac{\partial g}{\partial x_{3i+l}} \frac{\partial h}{\partial x_{3i+m}}.$$

In three dimensions this is simply the definition of the bracket given by Nambu [1].

Proposition 2.10. Consider a Nambu system $(M^{3n}, \omega^{(3)}, g, h)$. Let $f, g', h' \in C^\infty(M^{3n})$ satisfying $(dh \wedge dg) \sim (dh' \wedge dg')$ and $(df \wedge dg')^{\sharp\flat} = (df \wedge dg')$, then we have

$$\{f, g, h\}dg' = i_{(dg\wedge dh)^\sharp} i_{(df\wedge dg')^\sharp} \omega^{(3)}.$$

Proof. We have

$$\begin{aligned}\{f, g, h\}dg' &= (L_{(dg\wedge dh)^\sharp} f)dg' \\ &= (i_{(dg\wedge dh)^\sharp}(df \wedge dg')) \\ &= i_{(dg\wedge dh)^\sharp} i_{(df\wedge dg')^\sharp} \omega^{(3)}.\end{aligned}$$ □

The following proposition gives the relation between the bracket of 2-forms and the bracket of functions.

Proposition 2.11. Let $(M^{3n}, \omega^{(3)})$ be a Nambu manifold and let f, g, h_1, h_2 be C^∞ functions satisfying $(df \wedge dg)^{\sharp\flat} = df \wedge dg$ and $(dh_1 \wedge dh_2)^{\sharp\flat} = dh_1 \wedge dh_2$. Then

$$\{dh_1 \wedge dh_2, df \wedge dg\} = d\{f, h_1, h_2\} \wedge dg + df \wedge d\{g, h_1, h_2\}.$$

Proof. From proposition 2.7 we have

$$\{dh_1 \wedge dh_2, df \wedge dg\} = L_{(dh_1\wedge dh_2)^\sharp}(df \wedge dg).$$ □

Remark 2.13. If a function f is an integral of motion then the Nambu bracket of the function $\{f, \mathcal{H}_1, \mathcal{H}_2\}$ is zero and conversely. On the other hand, if β is a 2-form such that $\{d\mathcal{H}_1 \wedge d\mathcal{H}_2, \beta\} = 0$ then there exists a 2-form β' in the equivalence class of β which is an invariant of motion. Also by proposition 2.11 these two statements are consistent.

3. Applications of Nambu mechanics

The purpose of this section is to demonstrate that there are systems that can be described appropriately using the formalism developed here.

3.1. Coupled rigid bodies

We now consider the simplest case of a coupling between two symmetric tops. The coupling introduced is proportional to the z component of the angular momentum of each rotor (such an idealized situation corresponds under certain assumptions to the case of two symmetric tops interacting with each other through a magnetic moment coupling). The equations of motion for the angular momenta ($\{x_1, y_1, z_1\}$ and $\{x_2, y_2, z_2\}$) of the tops in their respective body coordinates are

$$\dot{x_1} = \frac{1}{I_{y_1} I_{z_1}}[y_1 z_1 (I_{z_1} - I_{y_1}) + I_{z_1} C_3 y_1 z_2]$$

$$\dot{y_1} = -\frac{1}{I_{x_1} I_{z_1}}[x_1 z_1 (I_{z_1} - I_{x_1}) + I_{z_1} C_3 x_1 z_2]$$

$$\dot{z_1} = 0$$

$$\dot{x_2} = \frac{1}{I_{y_2} I_{z_2}}[y_2 z_2 (I_{z_2} - I_{y_2}) + I_{z_2} C_3 y_2 z_1]$$

$$\dot{y_2} = -\frac{1}{I_{x_2} I_{z_2}}[x_2 z_2 (I_{z_2} - I_{x_2}) + I_{z_2} C_3 x_2 z_1]$$

$$\dot{z_2} = 0.$$

These equations have the generalized Nambu form in the sense of this paper and they can also be obtained from the following Nambu functions:

$$\mathcal{H}_1 = \tfrac{1}{2}({x_1}^2 + {y_1}^2 + {z_1}^2) + \tfrac{1}{2}({x_2}^2 + {y_2}^2 + {z_2}^2)$$

$$\mathcal{H}_2 = \frac{1}{2}\left(\frac{{x_1}^2}{I_{x_1}} + \frac{{y_1}^2}{I_{y_1}} + \frac{{z_1}^2}{I_{z_1}}\right) + \frac{1}{2}\left(\frac{{x_2}^2}{I_{x_2}} + \frac{{y_2}^2}{I_{y_2}} + \frac{{z_2}^2}{I_{z_2}}\right) + C_3 z_1 z_2.$$

It is obvious that the constant C_3 depends on the initial orientation of the tops in the laboratory frame. In the absence of coupling the tops obey the Euler equations individually. In the above vector field the terms like $I_{z_1} C_3 y_1 z_2$ can be considered to be the effective external torque on one top due to the presence of the other. The important point to note is that this torque merely changes the precession frequency of both the rigid bodies.

3.2. Fluid flows

It has been known for a long time that two-dimensional incompressible fluid flows can be studied using the two-dimensional Hamiltonian framework. Holm and Kimura [11] realized that the Nambu description is suitable for three-dimensional integrable flows of incompressible fluids in the Lagrangian picture. However, the three-dimensional Nambu system is not suitable as a framework for the formulation of non-integrable fluid flows. We suggest that this requirement can be fulfilled by an appropriate choice of $\mathcal{H}_1$, $\mathcal{H}_2$ in a $3n$-dimensional Nambu framework. Specifically we show that the Arter and Chandrashekhar flows (describing Rayleigh–Bénard convective motion) can be cast as flows on an invariant three-dimensional subspace of a six-dimensional Nambu system.

Consider a Nambu manifold $(\mathbb{R}^6, \omega^{(3)})$ where

$$\omega^{(3)} = dx \wedge dy \wedge dz + dx' \wedge dy' \wedge dz'$$

in standard coordinates $\{x, y, z, x', y', z'\}$ of $\mathbb{R}^6$.

It can be straightforwardly verified (using any symbolic manipulation package) that the Nambu functions

$$\mathcal{H}_1 = \log\left(\frac{\sin(x)}{\sin(y)}\right) - \log\left(\frac{\sin(x')}{\sin(y')}\right) - K^2\frac{\cos(x')}{\sin(x')}(y - y') - K^2\frac{\cos(y')}{\sin(y')}(x - x') \qquad (4)$$

$$\mathcal{H}_2 = \sin(x)\sin(y)\sin(z) - \sin(x')\sin(y')\sin(z') + (y - y')A + (x - x')B \qquad (5)$$

where

$$A = \frac{K^2 \cos(y') \cos(y) \sin^2(x) \sin(z)}{\cos(x) \sin(y') - K^2 \sin(x) \cos(y')}$$

$$B = -\frac{K^2 \cos(x') \cos(x) \sin^2(y) \sin(z)}{\cos(y) \sin(x') + K^2 \sin(y) \cos(x')}$$

produce a flow for which the subspace

$$x = x' \qquad y = y' \qquad z = z' \tag{6}$$

is an invariant subspace. Moreover, the flow restricted to this subspace has precisely the form of the Chandrashekhar flow. The equations governing Chandrashekhar flow are

$$\dot{x} = -\sin(x) \cos(y) \cos(z) - K^2 \cos(x) \sin(y) \cos(z)$$

$$\dot{y} = -\cos(x) \sin(y) \cos(z) + K^2 \cos(y) \sin(x) \cos(z)$$

$$\dot{z} = 2 \cos(x) \cos(y) \sin(z)$$

where (x, y, z) are the coordinates of the fluid particles in the Lagrangian picture.

In an exactly similar manner, if we choose the Nambu functions as

$$\mathcal{H}_1 = \log\left(\frac{\sin(x)}{\sin(y)}\right) - \log\left(\frac{\sin(x')}{\sin(y')}\right) - 2b\frac{\cos(y) \cos(2z')}{\sin(x') \cos(z)}(x - x')$$
$$+ 2b\frac{\cos(x) \cos(2z')}{\sin(y') \cos(z)}(y - y')$$

$$\mathcal{H}_2 = \sin(x) \sin(y) \sin(z) - \sin(x') \sin(y') \sin(z') - (x - x')C + (y - y')D$$

where

$$C = \frac{\cos(x) \sin(y)\Big((\cos^2(x) + \cos^2(y)) \cos(2z) \sin(z) - (\cos(2x) \cos(2y)) \sin(2z) \cos(z)\Big)}{\cos(2z)(\cos^2(x) - \cos^2(y))}$$

$$D = \frac{\cos(y) \sin(x)\Big((\cos^2(x) + \cos^2(y)) \cos(2z) \sin(z) - (\cos(2x) \cos(2y)) \sin(2z) \cos(z)\Big)}{\cos(2z)(\cos^2(x) - \cos^2(y))}$$

then in the invariant subspace we get equations governing Arter flow, namely

$$\dot{x} = -\sin(x) \cos(y) \cos(z) + b \cos(2x) \cos(2z)$$

$$\dot{y} = -\cos(x) \sin(y) \cos(z) + b \cos(2y) \cos(2z)$$

$$\dot{z} = 2 \cos(x) \cos(y) \sin(z) - b(\cos(2x) + \cos(2y)) \sin(2z)$$

where again (x, y, z) are coordinates in the Lagrangian picture.

4. Conclusions

We have developed a geometric framework for the formulation of generalized Nambu systems. This formalism is more suitable from the view point of dynamical systems. As demonstrated by the example of Arter flow, a potentially non-integrable flow finds a description in terms of generalized Nambu flow. Specifically, the Chandrashekhar flow and the Arter flow have been identified with the motion that takes place in an invariant three-dimensional subspace of a six-dimensional Nambu system.

An interesting feature of the present formalism is the following. Whereas a three bracket of functions gives rise to a non-associative structure, a Nambu Poisson bracket of 2-forms gives rise to a Lie algebra. It was found that formulations involving 2-forms provide a natural approach to a Nambu system of order three. We feel that it is worth investigating the issues such as symmetries, reduction and integrability for such systems further. Nambu systems of higher order could also be investigated. However, so far we have not carried this out due to the lack of appropriate physical examples.

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Appendix

Definition A.1 (block-diagonal form). Let $(M^{3n}, \omega^{(3)})$ be a Nambu manifold. A 2-form α is called a *block-diagonal form* if for some $X \in \mathcal{X}(M^{3n})$

$$i_X\omega^{(3)} = \alpha.$$

Definition A.2 (non-diagonal form). Let $(M^{3n}, \omega^{(3)})$ be a Nambu manifold. A 2-form α is called a *non-diagonal form* if for each $z_1, z_2 \in \mathcal{X}(M^{3n})$ such that $\alpha(z_1, z_2) \neq 0 \nexists X$ with the property $\omega^{(3)}(X, z_1, z_2) = \alpha(z_1, z_2)$.

Remark A.1. If a form is not non-diagonal this does not imply that it is block diagonal.

Proposition A.1. Let $(M^{3n}, \omega^{(3)})$ be a Nambu manifold and let α be any 2-form. Then

$$\alpha = \alpha^d + \alpha'$$

where α^d is a block-diagonal form and α' is a non-diagonal form, and the decomposition is unique.

Proof. Consider the Darboux coordinates

$$\begin{aligned}\alpha = \sum_{i,j=0}^{n-1}\sum_{l,m=1}^{3} \alpha_{3i+l\ \ 3j+m}\,dx^{3i+l} \wedge dx^{3j+m} \\ = \sum_{i=0}^{n-1}\sum_{l,m=1}^{3} \alpha_{3i+l\ \ 3i+m}\,dx^{3i+l} \wedge dx^{3i+m} \\ + \sum_{i,j=0,i\neq j}^{n-1}\sum_{l,m=1}^{3} \alpha_{3i+l\ \ 3j+m}\,dx^{3i+l} \wedge dx^{3j+m}.\end{aligned}$$

It is easy to see that the first term, which we denote by α^d, is a block-diagonal form and the second term, which we denote by α', is a non-diagonal form. Now we prove the uniqueness of the decomposition. Let α_1^d, α_1' and α_2^d, α_2' be two distinct decompositions of

α such that α_1^d, α_2^d are block-diagonal forms and α_1', α_2' are non-diagonal forms. Thus we have $\alpha_1^d + \alpha_1' = \alpha_2^d + \alpha_2'$: this implies that α_2' is not non-diagonal. Hence α has a unique decomposition. □

Definition A.3. We define a map $\sharp : \Omega_2^0(M^{3n}) \to \mathcal{X}(M^{3n}) : \alpha \mapsto \alpha^\sharp$ such that

$$i_{\alpha^\sharp}\omega^{(3)} = \alpha^d$$

where α^d is the block-diagonal part of α.

This map can be identified as the map introduced in section 2.2. Since α^d is unique the definition of the map is coordinate free.

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